Tonal Space Semantics

Mark Granroth-Wilding

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This document aims to describe a simple notation for the tonal space semantics. It is intented as a supplement to the journal paper, which covers the syntactic formalism in detail, for my second-year review document.

1 Tonic Semantics

The semantics of a tonic is a point in the tonal space. It is underspecified – it only specifies a point within an *enharmonic block* (see figure 1). It is therefore a coordinate between $\langle 0, 0\rangle$ and $\langle 3, 2\rangle$ and each coordinate denotes different infinite set of positions in the space.

A single tonic chord receives as its logical form a single-element list containing such a coordinate.

2 Cadence Semantics

The semantics of a cadence step is a predicate representing a movement in the tonic space. An extended cadence is interpreted as the recursive application of each movement to its resolution.

Authentic cadences – left steps – use the *leftonto* predicate and plagal cadences – right steps – the *rightonto* predicate.

We define *leftonto* (and likewise *rightonto*) as being subject to a reduction when applied to a list, as in the case of a tonic resolution, which has a single-element list as its semantics.

$$\mathit{leftonto}([X_0, X_1, ...]) \Rightarrow [\mathit{leftonto}(X_0), X_1, ...]$$

Example (1) shows an example of a two-step cadence – the familiar *IIm*^{7} *V*^{7} *I*. The derivation shows the combination of the semantics of each chord into the semantics for the sequence.

(1)

$$\begin{array}{ccc}
\mathrm{IIm}^7 & \mathrm{V}^7 & \mathrm{I} \\
\hline
\lambda x.\mathit{leftonto}(x) & \lambda x.\mathit{leftonto}(x) & [\langle 0,0\rangle] \\
\cline{2-3}
 & \multicolumn{2}{c}{[\mathit{lefonto}(\langle 0,0\rangle)]} \ > \\
\hline
\multicolumn{3}{c}{[\mathit{leftonto}(\mathit{leftonto}(\langle 0,0\rangle))]} \ >
\end{array}$$

The recursive application of multiple cadence steps can be combined ahead of time, before their application to their resolution, using the composition operator.

$$f \circ g \equiv \lambda x.f(g(x))$$

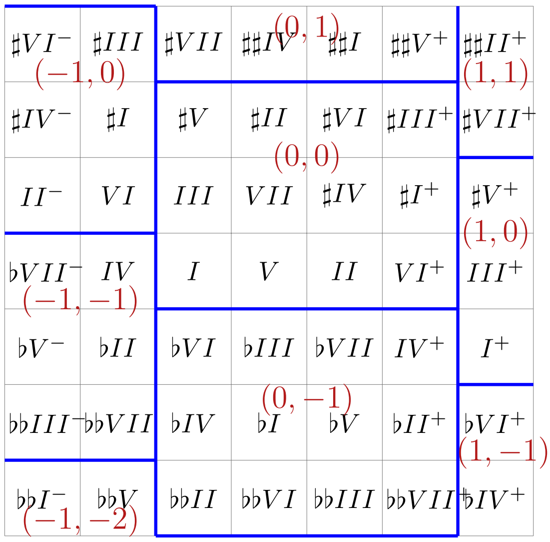

Figure 1: Enharmonic blocks at the centre of the space. Each position within these 4x3 blocks is equated by equal temperament with the same position within every other block.

3 Colouration Semantics

The lexicon includes some categories for interpreting colouration chords, which contribute nothing much to the functional structure of the harmony, but spice up the realisation a bit. Accordingly, these are given an empty semantics (that is, the identity function), which simply ignores them.

A typical example of this is the sequence I IV I, often played during long passages of a I chord. This is really a form of plagal cadence and a fine grained analysis might treat it as such. However, for most analysis purposes we wish to ignore this very brief excursion from the tonic. An example derivation using this empty semantics is shown in example (2).

(2)
$$\dfrac{\dfrac{\text{I}}{\lambda x.x} \quad \dfrac{\dfrac{\text{IV}}{\lambda x.x} \quad \dfrac{\text{I}}{[\langle 0,0\rangle]}}{[\langle 0,0\rangle]}>}{[\langle 0,0\rangle]}>$$

In many cases, we do not even return to the tonic after our excursion, continuing with a cadence straight after the IV. This is the purpose of the backward-facing colouration lexical category and the semantics ignores the IV in the same way.

4 Development Semantics

The development combinatory rule combines sequences of tonic passages and resolved cadences into larger units, ultimately into a whole piece of music. Every logical form introduced so far has been a single-item list, the behaviour of the development rule's semantics is rather trivial. It simply concatenates its two arguments: the syntax ensures these are lists.

Example (3) shows a pair of resolved cadences being combined in this way. Example (4) shows a tonic (a single-element list) combining with a subsequent resolved cadence.

(3)
$$\begin{array}{ccccc}
\text{IIm}^7 & \text{V}^7 & \text{I} & \text{V}^7 & \text{I} \\
\hline
\lambda x.\mathit{leftonto}(x) & \lambda x.\mathit{leftonto}(x) & [\langle 0,0\rangle] & \lambda x.\mathit{leftonto}(x) & [\langle 0,0\rangle)] \\
\cline{2-3}\cline{4-5}
 & \multicolumn{2}{c}{[\mathit{lefonto}(\langle 0,0\rangle)] \quad {\scriptstyle >}} & \multicolumn{2}{c}{[\mathit{lefonto}(\langle 0,0\rangle)] \quad {\scriptstyle >}} \\
\cline{1-3}
\multicolumn{3}{c}{[\mathit{leftonto}(\mathit{leftonto}(\langle 0,0\rangle))] \quad {\scriptstyle >}} & & \\
\hline
\multicolumn{5}{c}{[\mathit{leftonto}(\mathit{leftonto}(\langle 0,0\rangle)), \mathit{lefonto}(\langle 0,0\rangle)] \quad {\scriptstyle \mathsf{dev}}}
\end{array}$$

(4)
$$\begin{array}{cccc}
\text{I} & \text{IIm}^7 & \text{V}^7 & \text{I} \\
\hline
[\langle 0,0\rangle] & \lambda x.\mathit{leftonto}(x) & \lambda x.\mathit{leftonto}(x) & [\langle 0,0\rangle] \\
\cline{3-4}
 & & \multicolumn{2}{c}{[\mathit{lefonto}(\langle 0,0\rangle)] \quad {\scriptstyle >}} \\
\cline{2-4}
 & \multicolumn{3}{c}{[\mathit{leftonto}(\mathit{leftonto}(\langle 0,0\rangle))] \quad {\scriptstyle >}} \\
\hline
\multicolumn{4}{c}{[\langle 0,0\rangle, \mathit{leftonto}(\mathit{leftonto}(\langle 0,0\rangle))] \quad {\scriptstyle \mathsf{dev}}}
\end{array}$$

5 Coordination Semantics

Logical forms representing unresolved cadences can be *coordinated* to share their eventual resolution. This is carried out by the special musical *coordination* combinator. The semantics of this combinator simply conjoins the cadence logical forms using the $\wedge$ operator. Note that, unlike in language semantics, this conjunction operator must preserve the order of its arguments.

$$A \wedge B \not\equiv B \wedge A$$

We can also reduce brackets to reflect the associativity of the conjunction operator.

$$\begin{aligned} A \wedge B \wedge C &\equiv (A \wedge B) \wedge C \\ &\equiv A \wedge (B \wedge C) \end{aligned}$$

$$A \wedge (B \wedge C) \Rightarrow A \wedge B \wedge C$$

$$(A \wedge B) \wedge C \Rightarrow A \wedge B \wedge C$$

The functions that denote cadences are simply conjoined by $\wedge$:

(5)
$$\begin{array}{cc}
\text{IIm}^7\ \text{V}^7 & \text{IIm}^7\ \text{V}^7 \\
\hline
\lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) & \lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) \\
\hline
\multicolumn{2}{c}{\lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) \wedge \lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) \quad {\scriptstyle \&}}
\end{array}$$

The result is treated as a functor that can be applied to its resolution. It reduces under application to a list in the same way as *leftonto* and *rightonto*. Note that the individual cadences are not actually applied to the resolution at this stage.

(6)
$$
\begin{array}{lll}
\text{IIm}^7\ \text{V}^7 & \text{IIm}^7\ \text{V}^7 & \text{I} \\
\hline
\lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) & \lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) & [\langle 0,0\rangle] \\
\hline
\multicolumn{2}{l}{\lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) \wedge \lambda x.\mathit{leftonto}(\mathit{leftonto}(x))\ \&} & \\
\hline
\multicolumn{3}{l}{[(\lambda x.\mathit{leftonto}(\mathit{leftonto}(x)) \wedge \lambda x.\mathit{leftonto}(\mathit{leftonto}(x)))(\langle 0,0\rangle)]\ >}
\end{array}
$$

More than two cadences can be coordinated to share the same resolution. (The predicate *leftonto* is abbreviated to L to save space.)

(7)
$$
\begin{array}{llll}
\text{IIm}^7\ \text{V}^7 & \text{IIm}^7\ \text{V}^7 & \text{IIm}^7\ \text{V}^7 & \text{I} \\
\hline
\lambda x.L(L(x)) & \lambda x.L(L(x)) & \lambda x.L(L(x)) & [\langle 0,0\rangle] \\
\hline
\multicolumn{2}{l}{\lambda x.L(L(x)) \wedge \lambda x.L(L(x))\ \&} & & \\
\hline
\multicolumn{3}{l}{\lambda x.L(L(x)) \wedge \lambda x.L(L(x)) \wedge \lambda x.L(L(x))\ \&} & \\
\hline
\multicolumn{4}{l}{[(\lambda x.L(L(x)) \wedge \lambda x.L(L(x)) \wedge \lambda x.L(L(x)))(\langle 0,0\rangle)]\ >}
\end{array}
$$

The result of a coordination (once applied to its resolution) can become the recursive resolution of a prior cadence step.

(8)
$$
\begin{array}{llll}
\text{VI}^7 & \text{IIm}^7\ \text{V}^7 & \text{IIm}^7\ \text{V}^7 & \text{I} \\
\hline
\lambda x.L(x) & \lambda x.L(L(x)) & \lambda x.L(L(x)) & [\langle 0,0\rangle] \\
\hline
 & \multicolumn{2}{l}{\lambda x.L(L(x)) \wedge \lambda x.L(L(x))\ \&} & \\
\hline
 & \multicolumn{3}{l}{[(\lambda x.L(L(x)) \wedge \lambda x.L(L(x)))(\langle 0,0\rangle)]\ >} \\
\hline
\multicolumn{4}{l}{[L((\lambda x.L(L(x)) \wedge \lambda x.L(L(x)))(\langle 0,0\rangle))]\ >}
\end{array}
$$

However, this logical form will result in the same tonal space path as that which would have been produced by composing the VI^7 with the following IIm^7 V^7 before coordinating:

$$[(\lambda x.L(L(L(x))) \wedge \lambda x.L(L(x)))(\langle 0,0\rangle)]$$

We therefore define the following equivalence and by convention reduce the left-hand side form to the right-hand side wherever possible.

$$A\ ((B \wedge ...)\ C) \Rightarrow (A \circ B \wedge ...)\ C$$

6 Bigger Example

To show all of these things in action at once, the example below shows the compositional semantics being combined for a longer sequence. The sequence begins on a tonic, cadences back to the same tonic, with a *I IV I* colouration on the tonic. It then modulates, via a cadence, to a new key – the subdominant key. When it gets to its new tonic, there is another *IV* colouration, this time not returning to the tonic.

(9)

$$
\begin{array}{cccccccccccc}
\mathrm{I} & \mathrm{IIm}^7 & \mathrm{V}^7 & \mathrm{IIm}^7 & \mathrm{V}^7 & \mathrm{I} & \mathrm{IV} & \mathrm{I} & \mathrm{Vm}^7 & \mathrm{I}^7 & \mathrm{IV} & \flat\mathrm{VII}... \\
\hline
[\langle 0,0\rangle] & \lambda x.L(x) & \lambda x.L(x) & \lambda x.L(x) & \lambda x.L(x) & \lambda x.x & \lambda x.x & [\langle 0,0\rangle] & \lambda x.L(x) & \lambda x.L(x) & [\langle -1,0\rangle] & \lambda x.x \\
\cline{2-3}\cline{4-5}\cline{7-8}\cline{11-12}
& \multicolumn{2}{c}{\lambda x.L(L(x)) \;\; {\scriptstyle >\mathbf{B}}} & \multicolumn{2}{c}{\lambda x.L(L(x)) \;\; {\scriptstyle >\mathbf{B}}} & & \multicolumn{2}{c}{[\langle 0,0\rangle] \;\; {\scriptstyle >}} & & & \multicolumn{2}{c}{[\langle -1,0\rangle] \;\; {\scriptstyle <}} \\
\cline{2-5}\cline{6-8}\cline{10-12}
& \multicolumn{4}{c}{(\lambda x.L(L(x)) \wedge \lambda x.L(L(x))) \;\; {\scriptstyle \&}} & \multicolumn{3}{c}{[\langle 0,0\rangle] \;\; {\scriptstyle >}} & & \multicolumn{3}{c}{[L(\langle -1,0\rangle)] \;\; {\scriptstyle >}} \\
\cline{2-8}\cline{9-12}
& \multicolumn{7}{c}{[(\lambda x.L(L(x)) \wedge \lambda x.L(L(x)))(L(\langle 0,0\rangle))] \;\; {\scriptstyle >}} & \multicolumn{4}{c}{[L(L(\langle -1,0\rangle))] \;\; {\scriptstyle >}} \\
\cline{1-8}
\multicolumn{8}{c}{[\langle 0,0\rangle, (\lambda x.L(L(x)) \wedge \lambda x.L(L(x)))(L(\langle 0,0\rangle))] \;\; {\scriptstyle \mathbf{dev}}} & & & & \\
\hline
\multicolumn{12}{c}{[\langle 0,0\rangle, (\lambda x.L(L(x)) \wedge \lambda x.L(L(x)))(L(\langle 0,0\rangle)), L(L(\langle -1,0\rangle))] \;\; {\scriptstyle \mathbf{dev}}}
\end{array}
$$

7 Extracting the Tonal Space Path

The logical forms that come out of the above semantics represent paths through the tonal space. Although the tonic points are ambiguous in the representation, the points of a whole path are unambiguous, modulo an arbitrary choice of starting point. This is because certain constraints on the relative position of points must be satisfied: given the fully-specified position of the first point on the path, the rest of the points are constrained to individual points on the tonal space.

The most obvious constraint is on the point created by a left (or right) movement, denoted in the semantics by *leftonto* (or *rightonto*) predicates. In *leftonto*(*pth*), the point at which the movement begins must be one step in the grid to the right of the first point of *pth*. If the point (x, y) if fully specified, the whole path *leftonto*(*leftonto*((x, y))) is therefore also unambiguous.

Two cadences that share a resolution through coordination are constrained to end at the same point, since their points are constrained relative to their resolution.

There is no obvious constraint between items in the top-level list of tonics and resolved cadences. The most plausible choice of relative positions appears to be reached by constraining the start point of a particular item to be the closest possible point to the end point of the previous item.

For example, take the following two paths:

1. $[\langle 0, 0\rangle, \textit{leftonto}(\textit{leftonto}(\langle 0, 0\rangle))]$

2. $[\langle 0, 0\rangle, \textit{leftonto}(\textit{leftonto}(\textit{leftonto}(\langle 0, 0\rangle)))]$

The start of the second item in path 1 is dependent, ultimately, on the cadence resolution $\langle 0, 0\rangle$. But this point is underspecified: we can choose for it any of the infinite points that lie at $\langle 0, 0\rangle$ within their enharmonic block. Given an arbitrary choice of the first item's point at the central $(0, 0)$, we will choose the same point for the end of the second item, since it puts the start of the second item (now $(2, 0)$) as close as possible to $(0, 0)$.

In path 2, however, the second path begins at a point further from its ending. In this case we will choose $(-1, 1)$ as the start point for the second item by setting the $\langle 0, 0\rangle$ at its end to be at $(-4, 1)$.

Note that the choice of the first point on the path is unimportant: two paths identical in form, but occurring at different positions in the space can be considered equivalent, since the only difference between them is their absolute pitch and we (uncontraversially) consider precise absolute pitch not to be pertinent to musical semantics.

A simple algorithm can be constructed by means of a recursive transformation of the logical predicates to produce the flat tonal space path represented by a logical form generated by the grammar.